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LETTER TO THE EDITOR

Explicit solvability of a class of Maliuzhinets' equations with matrix coefficients

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Abstract. A coupled system of Maliuzhinets' functional equations for a pair of unknown functions is studied. It is shown that the system can be solved in an explicit form for a specific but non-trivial class of its matrix coefficients. Two examples arising in electromagnetic diffraction theory in a wedge-shaped region are briefly discussed.

The problem of diffraction by an impedance wedge has been solved by Maliuzhinets in his fundamental paper (Maliuzhinets 1958). The problem was reduced to the functional equations for one unknown function which was determined in an explicit form. Problems of modern diffraction theory in a wedge-shaped region with vector boundary conditions are reduced to the systems of functional equations for two unknown functions (see, for example, Bernard 1991, Lyalinov 1994). These equations cannot be decoupled in the general case. However, it is of great importance to have an exact solution in some practical situations.

The problem in question has the obvious similarity with the explicit commutative factorization of a 2×2 matrix (Chebotarev 1956, Khrapkov 1971, Daniele 1984, Hurd 1976, Rawlins and Williams 1981, Jones 1991 and many others). Recent progress is connected with the generalization on a class of the higher-order matrices (Luk'yanov 1983) and with studying its algebraic nature.

In this letter we demonstrate a class of vector Maliuzhinets' equations with 2×2 matrix coefficients which are solvable in an exact form. The matrix coefficients belong to a class of functionally commutative matrices $\mathbf{A}(\alpha)$ (Chebotarev 1956), i.e. $[\mathbf{A}(t), \mathbf{A}(\tau)] = 0$. We study the system of inhomogeneous Maliuzhinets' equations

$$\mathbf{A}_j(\alpha)\boldsymbol{\varphi}(\alpha - (-1)^j\Phi) - \mathbf{A}_j(-\alpha)\boldsymbol{\varphi}(-\alpha - (-1)^j\Phi) = \psi_j(\alpha) \qquad j = 1, 2 \tag{1}$$

where $\mathbf{A}_j(\alpha)$ is an entire 2×2 matrix, $\psi_j(\alpha) = (p_j(\alpha), q_j(\alpha))^{\mathrm{T}}$ is a given entire vector, $\Phi \in (0, \pi)$ and α is the complex variable. The unknown vector $\boldsymbol{\varphi}(\alpha) = (f(\alpha), g(\alpha))^{\mathrm{T}}$ is usually sought in the class $H_\delta(\Pi)$ of vectors which are regular in the strip $\Pi = \{\alpha : |\operatorname{Re}\alpha| < \Phi\}$ with the possible exception of one pole $\alpha_0(-\Phi < \alpha_0 < \Phi)$ and with a given value of $\operatorname{res}_{\alpha_0}\boldsymbol{\varphi}(\alpha)$ at α_0, and which are continuous in $\bar{\Pi}$. These vectors admit meromorphic continuation in $\mathbb{C}$. Moreover, the components $f(\alpha)$ and $g(\alpha)$ are bounded as $|\operatorname{Im}\alpha| \to \infty$ in Π and satisfy the conditions

$$f(\mathrm{i}\infty) = -f(-\mathrm{i}\infty) \qquad g(\mathrm{i}\infty) = -g(-\mathrm{i}\infty)$$
$$|f(\alpha) - f(\pm\mathrm{i}\infty)| < \text{constant} \times \exp(\pm\mathrm{i}\delta\alpha) \qquad \delta > 0$$
$$|g(\alpha) - g(\pm\mathrm{i}\infty)| < \text{constant} \times \exp(\pm\mathrm{i}\delta\alpha)$$

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as $\mathrm{Im}\,\alpha \to \pm\infty$ in Π. Usually the entries of $\mathbf{A}_j(\alpha)$ and $\psi_j(\alpha)$ are polynomials of $\sin\alpha$ and $\cos\alpha$ (Lyalinov 1994).

We construct the general solution of equations (1) with special coefficients for a class of vectors which includes $H_\delta(\Pi)$. First, for simplicity we assume that $\mathbf{A}_j(\alpha)$, $j = 1, 2$, have the form

$$\mathbf{A}_j(\alpha) = a_j(\alpha)\mathbf{I} + b_j(\alpha)\mathbf{J} \tag{2}$$

where

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{J} = \begin{pmatrix} 0 & 1 \\ \kappa & 0 \end{pmatrix} \qquad \mathbf{J}^2 = \kappa\mathbf{I}$$

$\kappa \in \mathbf{C}$, $a_j(\alpha)$ and $b_j(\alpha)$ are the entire functions. The matrices (2) are functionally commutative and admit an explicit solution of system (1). Furthermore, in this letter we demonstrate a more general class of matrices; however, the corresponding solution has a less compact form in comparison with that for the class (2). Equations (1) can be rewritten as follows

$$\varphi(\alpha - (-1^j)\Phi) = \mathbf{G}_j(\alpha)\varphi(-\alpha - (-1)^j\Phi) + \mathbf{A}_j^{-1}(\alpha)\psi_j(\alpha) \tag{3}$$

with

$$\mathbf{G}_j(\alpha) = \mathbf{A}_j^{-1}(\alpha)\mathbf{A}_j(-\alpha) \qquad j = 1, 2. \tag{4}$$

It is obvious that $\mathbf{G}_j(\alpha)\mathbf{G}_j(-\alpha) = \mathbf{I}$,

$$\mathbf{G}_j(\alpha) = u_j(\alpha)\mathbf{I} + v_j(\alpha)\mathbf{J} \tag{5}$$

$$u_j(\alpha) = \Delta_j^{-1}(\alpha)(a_j(\alpha)a_j(-\alpha) - \kappa b_j(\alpha)b_j(-\alpha))$$
$$v_j(\alpha) = \Delta_j^{-1}(\alpha)(a_j(\alpha)b_j(-\alpha) - b_j(\alpha)a_j(-\alpha))$$
$$\Delta_j(\alpha) = a_j^2(\alpha) - \kappa b_j^2(\alpha).$$

We adopt the first principal assumption that $\Delta_j(\alpha) = \det\mathbf{A}_j(\alpha)$, $j = 1, 2$, have no zeros on the imaginary axis $L = (-\mathrm{i}\infty, \mathrm{i}\infty)$. The second principal assumption is that $\log G_j(\alpha)$, $j = 1, 2$, can grow not faster than an exponent $\exp(\nu|\mathrm{Im}\,\alpha|)$, $\nu > 0$ as $|\mathrm{Im}\,\alpha| \to \infty$. The matrix (5) in system (3) can be represented in the exponential form (Khrapkov 1971)

$$\mathbf{G}_j(\alpha) = \sqrt{\lambda_j^1(\alpha)\lambda_j^2(\alpha)}\exp\left\{\frac{1}{2}\log\left(\frac{\lambda_j^1(\alpha)}{\lambda_j^2(\alpha)}\right)\frac{\mathbf{J}}{\sqrt{\kappa}}\right\}$$

where $\det\mathbf{G}_j(\alpha) = \lambda_j^1(\alpha)\lambda_j^2(\alpha) = u_j^2(\alpha) - \kappa v_j^2(\alpha)$, and $\lambda_j^1(\alpha) = u_j(\alpha) + \sqrt{\kappa}v_j(\alpha)$, $\lambda_j^2(\alpha) = u_j(\alpha) - \sqrt{\kappa}v_j(\alpha)$ are the eigenvalues of $\mathbf{G}_j(\alpha)$. As a result, we have

$$\log\mathbf{G}_j(\alpha) = \log(u_j^2(\alpha) - \kappa v_j^2(\alpha))\mathbf{I}/2 + \log\left(\frac{u_j(\alpha) + \sqrt{\kappa}v_j(\alpha)}{u_j(\alpha) - \sqrt{\kappa}v_j(\alpha)}\right)\mathbf{J}/(2\sqrt{\kappa}) \tag{6}$$

where $\log\mathbf{G}_j$ in (6) is fixed by the conditions $\log\mathbf{G}_j(0) = 0$, $\mathbf{G}_j(0) = \mathbf{I}$, provided that appropriate cuts are conducted on the α-plane. Note that, due to the first principal assumption, there are neither poles nor zeros on the imaginary axis L in the arguments of the logarithms in (6).

Let us define the *basic* matrix $\mathbf{X}(\alpha)$ which is a solution of the system

$$\mathbf{A}_j(\alpha)\mathbf{X}(\alpha - (-1)^j\Phi) = \mathbf{A}_j(-\alpha)\mathbf{X}(-\alpha - (-1)^j\Phi) \qquad j = 1, 2 \tag{7}$$

$\mathbf{X}(\alpha)$ and $\mathbf{X}^{-1}(\alpha)$ are regular in Π and are continuous in $\bar{\Pi}$ with $\log \mathbf{X}(\alpha)$ growing not faster than an exponent $\exp(\nu|\operatorname{Im}\alpha|)$ as $|\operatorname{Im}\alpha| \to \infty$. In accordance with the functional equations

$$\mathbf{X}(\alpha+2\Phi) = \mathbf{A}_j^{-1}(\alpha+\Phi)\mathbf{A}_j(-\alpha-\Phi)\mathbf{X}(-\alpha)$$
$$\mathbf{X}(\alpha-2\Phi) = \mathbf{A}_j^{-1}(\alpha-\Phi)\mathbf{A}_j(-\alpha+\Phi)\mathbf{X}(-\alpha)$$

$\mathbf{X}(\alpha)$ can be continued as a meromorphic matrix function on a wider strip $\{\alpha : |\operatorname{Re}\alpha| < 3\Phi\}$ and, then, in the same manner, on the whole α-plane. The matrices $\mathbf{G}_j(\alpha)$ in (5) are functionally commutative. Hence, from equations (7) we obtain

$$\mathbf{Y}(\alpha - (-1)^j\Phi) - \mathbf{Y}(-\alpha - (-1)^j\Phi) = \log \mathbf{G}_j(\alpha) \qquad j = 1, 2 \tag{8}$$

with $\mathbf{Y}(\alpha) = \log \mathbf{X}(\alpha)$ and $\log \mathbf{G}_j(\alpha)$ defined by expression (6). A solution of the inhomogeneous system (8) can be obtained by use of S-integrals (Tuzhilin 1973). Let us consider the integrals

$$S_j(\alpha) = \frac{\mathrm{i}}{8\Phi}\int_L \mathcal{F}(\tau)\sigma_j(\tau,\alpha)\,\mathrm{d}\alpha$$

where

$$\sigma_j(\tau,\alpha) = \frac{\sin\mu\tau}{\cos\mu\tau + (-1)^j\sin\mu\alpha} \qquad j = 1, 2$$

$\mu = \pi/2\Phi$, $\mathcal{F}(\tau)$ is an odd meromorphic function which decreases exponentially as $|\operatorname{Im}\tau| \to \infty$. The integral $S_1(\alpha)$ is regular in the strip $\Pi(-3\Phi,\Phi) = \{\alpha : -3\Phi < \operatorname{Re}\alpha < \Phi\}$ (Tuzhilin 1973). It can be continued on the strip $\Pi(-3\Phi, 5\Phi)$ by the formula

$$S_1(\alpha) = \frac{\mathrm{i}}{8\Phi}\int_L \frac{\mathcal{F}(\tau)\sin\mu\tau - \mathcal{F}(\alpha-\Phi)\sin\mu(\alpha-\Phi)}{\cos\mu\tau - \sin\mu\alpha} + \frac{(\alpha+\Phi)\mathcal{F}(\alpha-\Phi)}{4\Phi}.$$

The last formula enables us to verify that $S_1(\alpha)$ is a solution of the functional equations

$$S_1(\alpha+\Phi) - S_1(-\alpha+\Phi) = \mathcal{F}(\alpha)$$
$$S_1(\alpha-\Phi) - S_1(-\alpha-\Phi) = 0.$$

Let n and m be integer numbers such that $\log \mathbf{G}_1(\alpha)/\cos^n\mu\alpha$, $\log \mathbf{G}_2(\alpha)/\cos^m\mu\alpha (n\mu > \nu, m\mu > \nu)$ tend to zero exponentially as $|\operatorname{Im}\alpha| \to \infty$. By use of S-integrals the following lemma can be easily proved (see also Tuzhilin 1973).

Lemma. Let n, m and $\mathbf{G}_j(\alpha)$ satisfy the conditions described above, $\Delta_j(\alpha) \neq 0$ on L, then the general solution of equations (8), which is regular in Π, is continuous in $\bar{\Pi}$ and grows not faster than an exponent $\exp(\nu|\operatorname{Im}\alpha|)$, takes the form

$$\mathbf{Y}(\alpha) = \frac{\mathrm{i}\sin^n\mu\alpha}{8\Phi}\int_L \log \mathbf{G}_1(\tau)/\cos^n\mu\tau\,\sigma_1(\tau,\alpha)\,\mathrm{d}\tau - \frac{\mathrm{i}(-1)^m\sin^m\mu\alpha}{8\Phi} \times \int_L \log \mathbf{G}_2(\tau)/\cos^m\mu\tau\,\sigma_2(\tau,\alpha)\,\mathrm{d}\tau + \mathbf{Y}_0(\alpha) \qquad \alpha\in\Pi \tag{9}$$

where $\mathbf{Y}_0(\alpha)$ is a solution of the homogeneous equations (8): $\mathbf{Y}_0(\alpha)$ is a polynomial of $\sin\mu\alpha$ with constant matrix coefficients commuting with $\mathbf{J}$. Continuation of (9) on a wider strip can be carried out by use of functional equations (8) or with the aid of S-integrals (Tuzhilin 1973).

In construction of a basic solution $\mathbf{X}(\alpha)$ we put $\mathbf{Y}_0 \equiv 0$ and, after simple computations, find

$$\mathbf{X}(\alpha) = S(\alpha)\left\{\mathbf{I}\sinh T(\alpha) + \frac{\mathbf{J}}{\sqrt{\kappa}}\cosh T(\alpha)\right\} \qquad \alpha \in \Pi \tag{10}$$

where

$$S(\alpha) = \exp\left\{\frac{\mathrm{i}}{16\Phi}\sum_{j=1}^{2}\int_L \log(\lambda_j^1(\tau)\lambda_j^2(\tau))\frac{(-1)^{j+\delta_{2j}l_j+1}\sin\mu\tau\sin^{l_j}\mu\alpha\,\mathrm{d}\tau}{\cos^{l_j}\mu\tau(\cos\mu\tau+(-1)^j\sin\mu\alpha)}\right\}$$

$$T(\alpha) = \frac{\mathrm{i}}{16\Phi}\sum_{j=1}^{2}\int_L \log(\lambda_j^1(\tau)/\lambda_j^2(\tau))\frac{(-1)^{j+\delta_{2j}l_j+1}\sin\mu\tau\sin^{l_j}\mu\alpha\,\mathrm{d}\tau}{\cos^{l_j}\mu\tau(\cos\mu\tau+(-1)^j\sin\mu\alpha)}$$

where $l_j = n$ for $j = 1$ and $l_j = m$ for $j = 2$, δ_{ik} is the Kronecker symbol. By the direct substitution of formula (10) into (7) it is easily shown that $\mathbf{X}(\alpha)$ in (10) is the required basic solution. On any wider strip the expression for $\mathbf{X}(\alpha)$ is continued by use of the functional equations.

Having the basic solution, we introduce the new unknown vector φ_0 as follows

$$\varphi(\alpha) = \mathbf{X}(\alpha)\varphi_0(\alpha)$$

substitute $\varphi(\alpha)$ in equations (1) and obtain

$$\varphi_0(\alpha-(-1)^j\Phi) - \varphi_0(-\alpha-(-1)^j\Phi) = \mathbf{X}^{-1}(\alpha-(-1)^j\Phi)\mathbf{A}_j^{-1}(\alpha)\psi_j(\alpha) \tag{11}$$

with the right-hand side growing not faster than an exponent. The general solution is determined in the same manner as in the lemma. The solution of the homogeneous equations (11) can have poles in Π. For example, $\boldsymbol{\sigma}(\alpha) = \boldsymbol{C}/(\sin\mu\alpha - \sin\mu\alpha_0)$ is a solution of (11) with the trivial right-hand side. For the sake of compactness we omit the corresponding formulae.

The approach proposed in this letter can be easily generalized to the class of equations with

$$\mathbf{A}_j(\alpha) = a_j(\alpha)\mathbf{I} + b_j(\alpha)\mathbf{J}_+ + c_j(\alpha)\mathbf{J}_- \tag{12}$$

where

$$\mathbf{J}_+ = \begin{pmatrix} 0 & 1 \\ \kappa & \beta \end{pmatrix} \qquad \mathbf{J}_- = \begin{pmatrix} \beta & -1 \\ -\kappa & 0 \end{pmatrix}$$

$$\mathbf{J}_+^2 = \kappa\mathbf{I} + \beta\mathbf{J}_+ \qquad \mathbf{J}_-^2 = \kappa\mathbf{I} + \beta\mathbf{J}_-$$

$$\mathbf{J}_+\mathbf{J}_- = \mathbf{J}_-\mathbf{J}_+ = (-\kappa)\mathbf{I}.$$

Equations (1) with matrix (12) are exactly solvable provided that we have already constructed the basic matrix $\mathbf{X}(\alpha)$. The matrix $\mathbf{G}_j(\alpha)$ takes the form

$$\mathbf{G}_j(\alpha) = u_j(\alpha)\mathbf{I} + v_j(\alpha)\mathbf{J}_+ + w_j(\alpha)\mathbf{J}_-$$

with

$$u_j(\alpha) = (a_j(\alpha)a_j(-\alpha) - \kappa(c_j(\alpha)c_j(-\alpha) + b_j(\alpha)b_j(-\alpha) - c_j(\alpha)b_j(-\alpha) - c_j(-\alpha)b_j(\alpha)))/\tilde{\Delta}_j(\alpha)$$

$$v_j(\alpha) = (a_j(\alpha)b_j(-\alpha) + a_j(-\alpha)c_j(\alpha) + \beta c_j(\alpha)b_j(-\alpha))/\tilde{\Delta}_j(\alpha)$$

$$w_j(\alpha) = (a_j(-\alpha)b_j(\alpha) + a_j(\alpha)c_j(-\alpha) + \beta b_j(\alpha)c_j(-\alpha))/\tilde{\Delta}_j(\alpha)$$

$$\tilde{\Delta}_j(\alpha) = \det\mathbf{A}_j(\alpha)$$

and

$$\lambda_{1,2}^{j} = \frac{a_j(-\alpha) + \delta_{1,2} b_j(-\alpha) + \delta_{2,1} c_j(-\alpha)}{a_j(\alpha) + \delta_{1,2} b_j(\alpha) + \delta_{2,1} c_j(\alpha)} \qquad j = 1, 2$$

as its eigenvalues. The constants $\delta_{1,2} = \beta/2 \pm \sqrt{\beta^2/4 + \kappa}$ are the eigenvalues of the matrix J_+. If $\tilde{\Delta}_j(\alpha)$, $j = 1, 2$, have no zeros on L, we can compute $\log \mathbf{G}_j(\alpha)$ on L by the formula

$$\log \mathbf{G}_j = \frac{\log(\lambda_1^j/\lambda_2^j)}{\lambda_1^j - \lambda_2^j} \mathbf{G}_j + \frac{\lambda_1^j \log \lambda_2^j - \lambda_2^j \log \lambda_1^j}{\lambda_1^j - \lambda_2^j} \mathbf{I} \qquad \lambda_1^j \neq \lambda_2^j$$

$$(b_j(\tau) \neq c_j(\tau), \beta^2/4 + \kappa \neq 0)$$

which can be represented in the form (12). We exploit functional commutativity and reduce equations (7) to system (8). By use of S-integrals the basic matrix can be determined in the explicit form

$$\mathbf{X}(\alpha) = \exp\{\mathbf{I} \cdot S_0(\alpha) + \mathbf{J}_+ T_0(\alpha) + \mathbf{J}_- V_0(\alpha)\}. \tag{13}$$

For the sake of compactness we do not write down the expressions for S_0, T_0 and V_0. By use of the commutativity of $\mathbf{I}$, $\mathbf{J}_+$ and $\mathbf{J}_-$ the calculation of the exponent in (13) is equivalent to calculation of $\exp(T_0 \mathbf{J}_+)$ and $\exp(V_0 \mathbf{J}_-)$, which is a simple problem of linear algebra.

Consider two examples of coupled Maliuzhinets' equations arising in the theory of diffraction of an electromagnetic wave by a wedge. The first is from the problem of diffraction of a plane wave obliquely incident on the edge of an impedance wedge with identical unit relative surface impedances $W_j = W_j^{-1} = 1$. In this case, we have

$$\mathbf{A}_j(\alpha) = \begin{pmatrix} \sin\alpha - \sin\beta & W_0^{-1} \cos\alpha \cos\beta \\ -W_0 \cos\alpha \cos\beta & \sin\alpha - \sin\beta \end{pmatrix}$$

$$\psi_j = \begin{pmatrix} C_j \sin\alpha \\ D_j \sin\alpha \end{pmatrix} \qquad j = 1, 2$$

where W_0 and β are the parameters of the problem and C_j, D_j are constants. The second example of the matrices

$$\mathbf{A}_j(\alpha) = \begin{pmatrix} \sin\alpha + a_{11} & a_{12} \\ a_{21} & \sin\alpha + a_{22} \end{pmatrix} \qquad \psi_j = 0,\ j = 1, 2$$

can be faced with in the problem of diffraction by a wedge having identical anisotropic surface impedances on both sides (Lyalinov 1994). As a result, in both examples the corresponding problems can be solved in closed form. We hope that the proposed approach can be also used in different problems.

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